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# Zeros of Sobolev Orthogonal Polynomials of Gegenbauer Type

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Let $\{S_n\}_n$ denote the monic orthogonal polynomial sequence with respect to the Sobolev inner product

$$\langle f, g\rangle_S = \int f(x)\, g(x)\, d\psi_0(x) + \lambda \int f(x)\, g(x)\, d\psi_1(x),$$

where $\lambda > 0$ and $\{d\psi_0, d\psi_1\}$ is a so-called symmetrically coherent pair, with $d\psi_0$ or $d\psi_1$ the classical Gegenbauer measure $(x^2-1)^\alpha\, dx$, $\alpha > -1$. If $d\psi_1$ is the Gegenbauer measure, then $S_n$ has $n$ different, real zeros. If $d\psi_0$ is the Gegenbauer measure, then $S_n$ has at least $n-2$ different, real zeros. Under certain conditions $S_n$ has complex zeros. Also the location of the zeros of $S_n$ with respect to Gegenbauer polynomials, is studied. 

*Key Words:* Sobolev orthogonal polynomials; symmetrically coherent pairs; zeros; Gegenbauer polynomials.

## 1. INTRODUCTION

Consider the Sobolev inner product

$$\langle f, g\rangle_S = \int_a^b f(x)\, g(x)\, d\psi_0(x) + \lambda \int_a^b f'(x)\, g'(x)\, d\psi_1(x), \tag{1}$$

where $d\psi_0$ and $d\psi_1$ are measures on $(a, b)$ and $\lambda > 0$. Let $\{P_n\}_n$ and $\{Q_n\}_n$ denote monic orthogonal polynomial sequences (MOPS) with respect to $d\psi_0$ and $d\psi_1$, respectively. The pair $\{d\psi_0, d\psi_1\}$ is called a symmetrically coherent pair if there exist non-zero constants $D_n$ such that

$$Q_n = \frac{P'_{n+1}}{n+1} + D_n \frac{P'_{n-1}}{n-1}, \qquad n \geq 2. \tag{2}$$



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The concept of (symmetrically) coherent pairs was introduced by Iserles *et al.* in [3]. For a survey of results on (symmetrically) coherent pairs, see [4, 7]. In [8] all symmetrically coherent pairs have been determined by Meijer. Especially, it has been proved that one of the two measures $d\psi_0$ and $d\psi_1$ must be a Hermite or Gegenbauer measure.

In [5] Marcellán *et al.* investigated the coherent pair $\{d\psi_0, d\psi_1\} = \{x^\alpha e^{-x}\,dx, x^\alpha e^{-x}\,dx\}$, $\alpha > -1$ and $(a, b) = (0, \infty)$. They proved that the polynomial $S_n$ has $n$ different, real zeros. The same authors proved a similar result in [6] for the symmetrically coherent pair $\{(1-x^2)^\alpha, (1-x^2)^\alpha\}$, $\alpha > -1$, which is a special case of what is called "Type D" in this paper. In [2] De Bruin and Meijer proved that all polynomials $S_n$ following from coherent pairs, have $n$ different, real zeros.

Symmetrically coherent pairs of Hermite type has been studied in [1]. In that case it has been proved that if $d\psi_1$ is the Hermite measure $e^{-x^2}\,dx$, then $S_n$ has $n$ different, real zeros. If $d\psi_0$ is the Hermite measure, then $S_n$ has at least $n-2$ different, real zeros and under certain conditions $S_n$ has complex zeros.

The aim of this paper is to determine the location of the zeros of the Sobolev polynomials $S_n$, where $\{S_n\}_n$ is the MOPS with respect to the inner product (1) and where $(a, b) = (-1, 1)$. We assume $\{d\psi_0, d\psi_1\}$ to be a symmetrically coherent pair of Gegenbauer type.

In Section 2 we recall some well known properties of Gegenbauer polynomials, which will be used in this paper. In Section 3 we divide the symmetrically coherent pairs of Gegenbauer type in five classes (type A, B, C, D and E) and we determine general properties which hold for all the classes. In Section 4 we introduce moments and we determine the sign of these moments. In Section 5 we will use the moments and Gauss quadrature to determine the location of the zeros of the polynomials $S_n$. Polynomials $S_n$ following from symmetrically coherent pairs of type A, B, C and D turn out to have $n$ different, real zeros. For type E the polynomials $S_n$ have at least $n-2$ different, real zeros. Moreover, if $n = 2m+1$, $S_n$ has $n$ different, real zeros. In Section 5.1 we study the case where $\lambda \to \infty$ for type E and prove that, under certain conditions, $S_{2m}$ has complex zeros.

Since $d\psi_0$ for the Gegenbauer type in most of the cases depends on a parameter $\alpha$, we will sometimes use a subscript $\alpha$ to be able to distinguish two different measures $d\psi_0$ with a different parameter $\alpha$: $d\psi_{0,\alpha}$. Throughout this paper we will use the following notations:

$$\langle f, g\rangle_{i,\alpha} = \int_{-1}^{1} f(x)\, g(x)\, d\psi_{i,\alpha}(x), \qquad i = 0, 1$$

$$\langle f, g\rangle_{S,\alpha} = \int_{-1}^{1} f(x)\, g(x)\, d\psi_{0,\alpha}(x) + \lambda \int_{-1}^{1} f'(x)\, g'(x)\, d\psi_{1,\alpha}(x)$$

$$\|f\|_{i,\alpha}^2 = \int_{-1}^{1} f^2(x)\, d\psi_{i,\alpha}(x), \qquad i = 0, 1$$

$$\|f\|_{S,\alpha}^2 = \int_{-1}^{1} f^2(x)\, d\psi_{0,\alpha}(x) + \lambda \int_{-1}^{1} (f'(x))^2\, d\psi_{1,\alpha}(x)$$

When there can be no confusion we will omit the subscript $\alpha$.

## 2. CLASSICAL GEGENBAUER POLYNOMIALS

Let $\{G_n^{(\alpha)}\}_n$ be the sequence of *monic* Gegenbauer polynomials with parameter $\alpha > -1$. For the classical Gegenbauer polynomials the following properties are known (see [10]):

$G_n^{(\alpha)}$ is orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)(1-x^2)^{\alpha}\, dx. \tag{3}$$

The following relations are satisfied

$$\frac{d}{dx} G_n^{(\alpha)}(x) = n G_{n-1}^{(\alpha+1)}(x), \tag{4}$$

$$\frac{d}{dx} \left\{ G_{n+1}^{(\alpha)}(x) - \frac{n(n+1)}{4(n+\alpha+\frac{1}{2})(n+\alpha-\frac{1}{2})} G_{n-1}^{(\alpha)}(x) \right\} = (n+1)\, G_n^{(\alpha)}(x). \tag{5}$$

The three-term recurrence relation reads

$$G_{n+1}^{(\alpha)}(x) = x G_n^{(\alpha)}(x) - \frac{n(n+2\alpha)}{(2n+2\alpha-1)(2n+2\alpha+1)} G_{n-1}^{(\alpha)}(x). \tag{6}$$

For $G_n^{(\alpha)}(1)$ we have

$$G_n^{(\alpha)}(1) = \frac{\Gamma(\alpha+\frac{1}{2})\, \Gamma(n+2\alpha+1)}{2^n \Gamma(2\alpha+1)\, \Gamma(n+\alpha+\frac{1}{2})}. \tag{7}$$

From (3) follows that

$$G_n^{(\alpha)}(x) = P_n^{(\alpha,\alpha)}(x),$$

where $P_n^{(\alpha,\beta)}(x)$ is the *monic* Jacobi polynomial of degree $n$, orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)(1-x)^{\alpha}\, (1+x)^{\beta}\, dx.$$

The polynomials $G_n^{(\alpha)}$ and $P_n^{(\alpha,\beta)}$ satisfy the following relations

$$G_{2m}^{(\alpha)}(x)=2^{-m}P_m^{(\alpha,-\frac{1}{2})}(2x^2-1), \qquad G_{2m+1}^{(\alpha)}(x)=2^{-m}xP_m^{(\alpha,\frac{1}{2})}(2x^2-1). \quad (8)$$

The Rodrigues formula for monic Jacobi polynomials reads

$$P_n^{(\alpha,\beta)}(x)=(-1)^n\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}(1-x)^{-\alpha}(1+x)^{-\beta}\times$$
$$\times\left(\frac{d}{dx}\right)^n[(1-x)^{n+\alpha}(1+x)^{n+\beta}]. \quad (9)$$

## 3. SYMMETRICALLY COHERENT PAIRS OF GEGENBAUER TYPE

In this section we will determine some properties of symmetrically coherent pairs of Gegenbauer type, that will be useful later on in this paper.

Consider the inner product

$$\langle f,g\rangle_S=\int_{-1}^{1}f(x)\,g(x)\,d\psi_0(x)+\lambda\int_{-\infty}^{\infty}f'(x)\,g'(x)\,d\psi_1(x), \quad (10)$$

where $d\psi_0$ is a measure on $(-1,1)$, $d\psi_1$ is a measure on $\mathbb{R}$ with a continuous part only on $(-1,1)$ and $\lambda>0$. Let $\{P_n^\alpha\}_n$ denote the MOPS with respect to $\langle .,.\rangle_0$ and $\{Q_n^\alpha\}_n$ the MOPS with respect to $\langle .,.\rangle_1$. We assume $\{d\psi_0, d\psi_1\}$ to be a symmetrically coherent pair, i.e. there exist non-zero constants $D_n$ such that

$$Q_n^\alpha=\frac{(P_{n+1}^\alpha)'}{n+1}+D_n\frac{(P_{n-1}^\alpha)'}{n-1}, \qquad n\geqslant 2. \quad (11)$$

In [8] Meijer proved that there exist five types of symmetrically coherent pairs $\{d\psi_0, d\psi_1\}$ with $d\psi_0$ or $d\psi_1$ the classical Gegenbauer measure:

A $\quad \{(x^2+\xi^2)(1-x^2)^{\alpha-1}\,dx,\,(1-x^2)^\alpha\,dx\}, \qquad \alpha>0,$

B $\quad \{(\xi^2-x^2)(1-x^2)^{\alpha-1}\,dx,\,(1-x^2)^\alpha\,dx\}, \qquad |\xi|>1, \alpha>0,$

C $\quad \{dx+M\delta(-1)+M\delta(1),\,dx\}, \qquad M\geqslant 0,$

D $\quad \left\{(1-x^2)^{\alpha-1}\,dx,\,\frac{(1-x^2)^\alpha}{\xi^2-x^2}\,dx+M\delta(-\xi)+M\delta(\xi)\right\},$

$|\xi|>1, \alpha>0, M\geqslant 0,$

E $\quad \left\{(1-x^2)^{\alpha-1}\,dx, \frac{(1-x^2)^\alpha}{x^2+\xi^2}\,dx\right\}, \qquad \xi \neq 0, \alpha > 0.$

For these five types we can determine the constants $D_n$ in (11). We will denote $D_n$ for type A by $D_n^A$, etc. We use here and elsewhere in this paper, implicitly, that if $n$ is odd/even, the polynomials $P_n^\alpha$, $Q_n^\alpha$ and $S_n$ are odd/even. Also we will use implicitly that the integral, if convergent, over a symmetric interval of an odd function equals zero.

Type A. In this case the polynomials $Q_n^\alpha$ are the Gegenbauer polynomials $G_n^{(\alpha)}$ and for $D_n$ we obtain, by expanding $G_{n+1}^{(\alpha-1)}$ in terms of $P_i^\alpha$, using the orthogonality of $G_{n+1}^{(\alpha-1)}$ and using (4),

$$D_n^A = \frac{n-1}{n+1} \frac{\|G_{n+1}^{(\alpha-1)}\|_{1,\alpha-1}^2}{\|P_{n-1}^\alpha\|_{0,\alpha}^2} > 0.$$

Type B. In the same way as for type A we find

$$D_n^B = -\frac{n-1}{n+1} \frac{\|G_{n+1}^{(\alpha-1)}\|_{1,\alpha-1}^2}{\|P_{n-1}^\alpha\|_{0,\alpha}^2} < 0.$$

Type C. In this case the polynomials $Q_n$ are the Legendre polynomials $G_n^{(0)}$. We expand $G_{n+1}^{(0)} - \frac{n(n+1)}{4(n+1/2)(n-1/2)} G_{n-1}^{(0)}$ in terms of $P_i$. From (7) follows that $G_{n+1}^{(0)}(1) = \frac{n(n+1)}{4(n+1/2)(n-1/2)} G_{n-1}^{(0)}(1)$ and using (5) we obtain

$$D_n^C = -\frac{n(n-1)}{4(n+\frac{1}{2})(n-\frac{1}{2})} \frac{\|G_{n-1}^{(0)}\|_1^2}{\|P_{n-1}\|_0^2} < 0.$$

Type D. In this case the polynomials $P_n^\alpha$ are the Gegenbauer polynomials $G_n^{(\alpha-1)}$ and for $D_n$ we obtain, by expanding $Q_n^\alpha$ in terms of $G_i^{(\alpha)}$, using the orthogonality of $Q_n^\alpha$ and using (4),

$$D_n^D = -\frac{\|Q_n^\alpha\|_{1,\alpha}^2}{\|G_{n-2}^{(\alpha)}\|_{0,\alpha+1}^2} < 0.$$

Type E. In the same way as for type D, we find

$$D_n^E = \frac{\|Q_n^\alpha\|_{1,\alpha}^2}{\|G_{n-2}^{(\alpha)}\|_{0,\alpha+1}^2} > 0.$$

Let $\{S_n^\lambda\}_n$ be the MOPS with respect to the inner product (10). When we do not explicitly need the value of $\lambda$, we will omit the superscript $\lambda$.

LEMMA 3.1. *For the polynomials $P_n^\alpha$ and $S_n$ we have the following relation*

$$\frac{P_{n+1}^\alpha}{n+1}+D_n\frac{P_{n-1}^\alpha}{n-1}=\frac{S_{n+1}}{n+1}+d_n\frac{S_{n-1}}{n-1}, \qquad n\geqslant 2, \tag{12}$$

*and for the polynomials $Q_n$ and $S_n$ we have*

$$Q_n^\alpha=\frac{S'_{n+1}}{n+1}+d_n\frac{S'_{n-1}}{n-1}, \qquad n\geqslant 2, \tag{13}$$

*where*

$$d_n=D_n\frac{\|P_{n-1}^\alpha\|_{0,\alpha}^2}{\|S_{n-1}\|_{S,\alpha}^2}. \tag{14}$$

*Proof.* Because $\{d\psi_0, d\psi_1\}$ is a symmetrically coherent pair, $P_n^\alpha$ and $Q_n^\alpha$ satisfy relation (11). We expand $P_{n+1}^\alpha/(n+1)+D_n(P_{n-1}^\alpha/(n-1))$ in terms of $S_i$:

$$\frac{P_{n+1}^\alpha}{n+1}+D_n\frac{P_{n-1}^\alpha}{n-1}=\frac{S_{n+1}}{n+1}+\sum_{i=0}^{n}c_{n,i}S_i.$$

Using the orthogonality of $P_n^\alpha$ and $Q_n^\alpha$ we find

$$c_{n,i}=\begin{cases}0 & \text{if } i\leqslant n-2,\\ \dfrac{D_n}{n-1}\dfrac{\|P_{n-1}^\alpha\|_{0,\alpha}^2}{\|S_{n-1}\|_{S,\alpha}^2} & \text{if } i=n-1,\\ 0 & \text{if } i=n,\end{cases}$$

which gives

$$\frac{P_{n+1}^\alpha}{n+1}+D_n\frac{P_{n-1}^\alpha}{n-1}=\frac{S_{n+1}}{n+1}+d_n\frac{S_{n-1}}{n-1}, \tag{15}$$

with

$$d_n=D_n\frac{\|P_{n-1}^\alpha\|_{0,\alpha}^2}{\|S_{n-1}\|_{S,\alpha}^2}.$$

By differentiating (15) the lemma follows. ∎

*Remark* 3.1. We point out the fact that the sign of $d_n$ depends on the sign of $D_n$, i.e.

$$\operatorname{sgn} d_n = \operatorname{sgn} D_n.$$

Later on we will use the notation $d_n^A$ to denote the constants $d_n$ for type A, etc.

We define the polynomials $S_n^\infty$ by

$$S_n^\infty(x) = \lim_{\lambda \to \infty} S_n^\lambda(x).$$

Since $S_0^\lambda(x) = 1$, $S_1^\lambda(x) = x$ and $S_2^\lambda(x) = P_2^\alpha(x)$, $S_n^\infty$ exists for $n \leqslant 2$. The following lemma shows that $S_n^\infty$ also exists for $n \geqslant 3$.

LEMMA 3.2. *For $n \geqslant 3$ we have*

$$S_n^\infty(x) = P_n^\alpha(x) + D_{n-1} \frac{n}{n-2} P_{n-2}^\alpha(x).$$

*Proof.* We will use (15) and (14). Observe that for $n \geqslant 1$

$$\|S_{n-1}^\lambda\|_{S,\alpha}^2 > \lambda \int_{-1}^{1} ((S_{n-1}^\lambda)'(x))^2 \, d\psi_1(x),$$

and with the extremal property of the norm of orthogonal polynomials

$$\lambda \int_{-1}^{1} ((S_{n-1}^\lambda)'(x))^2 \, d\psi_1 > \lambda (n-1)^2 \int_{-1}^{1} (Q_{n-2}^\alpha(x))^2 \, d\psi_1.$$

Using (14) we then have

$$\lim_{\lambda \to \infty} d_n = 0, \qquad n \geqslant 2.$$

Using that $S_1(x) = x$ and (12) we find

$$S_3^\infty(x) = P_3^\alpha(x) + 3 D_2 P_1^\alpha(x).$$

Using that $S_2(x) = P_2^\alpha(x)$ and (12) we have

$$S_4^\infty(x) = P_4^\alpha(x) + 2 D_3 P_2^\alpha(x).$$

Now the lemma follows by induction. ∎

*Remark* 3.2. Since $Q_0^\alpha(x) = 1$ and $Q_1^\alpha(x) = x$, differentiating $S_n^\infty$, Lemma 3.2 and (11) give

$$(S_n^\infty)'(x) = n Q_{n-1}^\alpha(x), \qquad n \geqslant 1. \tag{16}$$

## 4. THE MOMENTS

In this section we introduce moments for each of the five types of symmetrically coherent pairs mentioned in Section 3. Using these moments we find lemmas, which we will use to determine the location of the zeros of $S_n$.

We distinguish two cases: $n$ is even or $n$ is odd.

**n=2m.** Type A. For $m \geqslant 1$ and $i = -2, -1, ..., m-1$ we define the moments $m^m_{i,A}$ by

$$m^m_{i,A} = \int_{-1}^{1} S_{2m}(x)(x^2+\xi^2)^{i+1}(1-x^2)^{\alpha-1}\,dx, \qquad i \geqslant -1,$$

$$m^m_{-2,A} = 0.$$

Type B. For $m \geqslant 1$ and $i = -2, -1, ..., m-1$ we define the moments $m^m_{i,B}$ by

$$m^m_{i,B} = \int_{-1}^{1} S_{2m}(x)(x^2-\xi^2)^{i+1}(1-x^2)^{\alpha-1}\,dx, \qquad i \geqslant -1,$$

$$m^m_{-2,B} = 0.$$

Type C. For $m \geqslant 1$ and $i = 0, 1, ..., m-1$ we define the moments $m^m_{i,C}$ by

$$m^m_{i,C} = \int_{-1}^{1} S_{2m}(x)(x^2-1)^{i}\,dx, \qquad i \geqslant 0.$$

Type D. For $m \geqslant 1$ and $i = -1, 0, ..., m-1$ we define the moments $m^m_{i,D}$ by

$$m^m_{i,D} = \int_{-1}^{1} S_{2m}(x)(x^2-\xi^2)^{i}(1-x^2)^{\alpha-1}\,dx, \qquad i \geqslant 0,$$

$$m^m_{-1,D} = 0.$$

Type E. For $m \geqslant 1$ and $i = -1, 0, ..., m-1$ we define the moments $m^m_{i,E}$ by

$$m^m_{i,E} = \int_{-1}^{1} S_{2m}(x)(x^2+\xi^2)^{i}(1-x^2)^{\alpha-1}\,dx, \qquad i \geqslant 0,$$

$$m^m_{-1,E} = 0.$$

LEMMA 4.1. *For $m \geqslant 2$ and for $1 \leqslant i \leqslant m-1$ we have*

$$m^m_{i,A} = -2\lambda i\{a^A_i m^m_{i-1,A} - b^A_i m^m_{i-2,A} + (i-1)\, c^A_i m^m_{i-3,A}\},$$
$$m^m_{i,B} = 2\lambda i\{a^B_i m^m_{i-1,B} + b^B_i m^m_{i-2,B} + (i-1)\, c^B_i m^m_{i-3,B}\},$$

*and for $m \geqslant 3$ and $2 \leqslant i \leqslant m-1$ we have*

$$m^m_{i,C} = 2\lambda i\{(2i-1)\, m^m_{i-1,C} + 2(i-1)\, m^m_{i-2,C}\},$$
$$m^m_{i,D} = 2\lambda i\{a^D_i m^m_{i-1,D} + b^D_i m^m_{i-2,D} + (i-2)\, c^D_i m^m_{i-3,D}\},$$
$$m^m_{i,E} = -2\lambda i\{a^E_i m^m_{i-1,E} - b^E_i m^m_{i-2,E} + (i-2)\, c^E_i m^m_{i-3,E}\},$$

*where $a^j_i > 0$, $b^j_i > 0$ and $c^j_i \geqslant 0$, $j = A, B, D, E$.*

*Proof.* For type A we have for $1 \leqslant i \leqslant m-1$, $\langle S_{2m}, (x^2+\xi^2)^i\rangle_S = 0$, hence

$$m^m_{i,A} = -2\lambda i \int_{-1}^{1} xS'_{2m}(x)(x^2+\xi^2)^{i-1}\,(1-x^2)^\alpha\, dx.$$

Using integration by parts we then find

$$a^A_i = 2i + 2\alpha - 1$$
$$b^A_i = 2i - 1 + \xi^2\,(4i + 2\alpha - 3)$$
$$c^A_i = 2\xi^2\,(1+\xi^2).$$

In the same way we find for type D:

$$a^D_i = 2i + 2\alpha - 3$$
$$b^D_i = \xi^2\,(4i + 2\alpha - 7) - 2i + 3$$
$$c^D_i = 2\xi^2\,(\xi^2 - 1).$$

Observe that by replacing $\xi^2$ in $m^m_{i,A}$ by $-\xi^2$, we get the result for type B. In the same way we can obtain the result for type E, from type D. We get the result for type C from type B, by substituting $\xi^2 = 1$ and $\alpha = 0$. ▌

LEMMA 4.2. *For $m \geqslant 2$ we have*

(i) $m^m_{0,j} = 0, \qquad j = A, B, D, E,$

$m^m_{0,C} \leqslant 0,$

(ii) $\operatorname{sgn} m^m_{i,A} = (-1)^{m+i+1}, \qquad i = -1, 1, 2, ..., m-1,$

$m^m_{i,B} > 0, \qquad i = -1, 1, 2, ..., m-1,$

$m^m_{i,C} < 0, \qquad i = 1, 2, ..., m-1,$

$m^m_{i,D} < 0, \qquad i = 1, 2, ..., m-1,$

$\operatorname{sgn} m^m_{i,E} = (-1)^{m+i+1}, \qquad i = 1, 2, ..., m-1.$

*Proof.* From $\langle S_{2m}, 1\rangle_S = 0$ we obtain $m^m_{0,j} = 0$ for $j = A, B, D, E$ and

$$m^m_{0,C} = -2MS_{2m}(1). \tag{17}$$

Using Lemma 4.1 we get for $i = 1$

$$m^m_{1,A} = 2\lambda b^A_1 m^m_{-1,A},$$

hence $\operatorname{sgn} m^m_{1,A} = \operatorname{sgn} m^m_{-1,A}$.

From $\langle S_{2m}, x^2+\xi^2\rangle_S = 0$ we obtain

$$m^m_{1,A} = -2\lambda \int_{-1}^{1} S'_{2m}(x)\, x(1-x^2)^\alpha\, dx.$$

Using $\langle G^{(\alpha)}_{2m-1}, x\rangle_1 = 0$ and applying (13) repeatedly, we get

$$\int_{-1}^{1} \frac{S'_{2m}(x)}{2m} x(1-x^2)^\alpha\, dx$$
$$= (-1)^{m-1}\, d^A_{2m-1} d^A_{2m-3} \cdots d^A_3 \int_{-1}^{1} \frac{S'_2(x)}{2} x(1-x^2)^\alpha\, dx.$$

Since $\int_{-1}^{1} (S'_2(x)/2)\, x(1-x^2)^\alpha\, dx = \int_{-1}^{1} x^2\, (1-x^2)^\alpha\, dx > 0$ and $d^A_i > 0$ we get

$$\operatorname{sgn} m^m_{-1,A} = \operatorname{sgn} m^m_{1,A} = (-1)^m.$$

By Lemma 4.1 and induction the lemma follows for type A. The proof for types B, D, and E runs along the same lines.

For type C we obtain in the same way

$$m^m_{1,C} = -2\lambda \int_{-1}^{1} xS'_{2m}(x)\, dx < 0.$$

Integration by parts and (17) give

$$m^m_{1,C} = -2\lambda(2S_{2m}(1) - m^m_{0,C}) = -4\lambda(1+M)\, S_{2m}(1).$$

From this follows that $S_{2m}(1) > 0$ and therefore $m_{0,C}^m \leqslant 0$. Then by induction the lemma follows for type C. ∎

Let $\pi(x)$ denote an even monic polynomial of degree $2k$, such that all zeros of $\pi(x)$ are real. For each type $j$ we define an integral $I_{1,l}^j$ by

$$I_{1,l}^j = \int_{-1}^{1} S_{2m}(x)\,\pi(x)(x^2+\xi^2)^l\,(1-x^2)^{\alpha-1}\,dx, \qquad j = A, E,$$

$$I_{1,l}^j = \int_{-1}^{1} S_{2m}(x)\,\pi(x)(x^2-\xi^2)^l\,(1-x^2)^{\alpha-1}\,dx, \qquad j = B, D,$$

$$I_{1,l}^C = \int_{-1}^{1} S_{2m}(x)\,\pi(x)(x^2-1)^l\,dx.$$

LEMMA 4.3. *Let* $m \geqslant 2$.

(i) *If* $1 \leqslant k+l \leqslant m$ *and* $\pi(x) \neq x^2$, *then* $\operatorname{sgn} I_{1,l}^A = (-1)^{m+k+l}$.

(ii) *If* $1 \leqslant k+l \leqslant m$, $\pi(x) \neq x^2-\xi^2$ *and if all zeros of* $\pi(x)$ *lie in the interval* $[-|\xi|, |\xi|]$, *then* $I_{1,l}^B > 0$.

(iii) *If* $1 \leqslant k+l \leqslant m-1$ *and all zeros of* $\pi(x)$ *lie in the interval* $[-1, 1]$, *then* $I_{1,l}^C < 0$.

(vi) *If* $1 \leqslant k+l \leqslant m-1$ *and all zeros of* $\pi(x)$ *lie in the interval* $[-|\xi|, |\xi|]$, *then* $I_{1,l}^D < 0$.

(v) *If* $1 \leqslant k+l \leqslant m-1$, *then* $\operatorname{sgn} I_{1,l}^E = (-1)^{m+k+l+1}$.

*Proof.* We proof the lemma for type A. For the other types the proof is similar.

Let $x_1, ..., x_k$ denote the non-negative zeros of $\pi(x)$. For $i = 1, ..., k$ put $t_i^2 = x_i^2 + \xi^2$. Then

$$\pi(x) = (x^2+\xi^2-t_1^2)\cdots(x^2+\xi^2-t_k^2)$$
$$= \sum_{i=0}^{k} c_i(x^2+\xi^2)^i,$$

where $c_k = 1$ and if $c_i \neq 0$, then $\operatorname{sgn} c_i = (-1)^{k-i}$. Then

$$I_{1,l}^A = \sum_{i=0}^{k} c_i m_{i+l-1,A}^m.$$

Using Lemma 4.2 we obtain that if $c_i \neq 0$ and $i \neq 1$, then $\operatorname{sgn} c_i m_{i+l-1,A}^m = (-1)^{m+k+l}$. Hence $\operatorname{sgn} I_{1,l}^A = (-1)^{m+k+l}$. Notice that if $\xi = 0$ and $\pi(x) = x^2$, then $c_0 = 0$, which gives $I_{1,0}^A = 0$.

For type B observe that, since the zeros of $\pi(x)$ lie in the interval $[-|\xi|, |\xi|]$, $t_i = x_i^2 - \xi^2$ is non-positive, therefore $c_i \geqslant 0$. ∎

**n=2m+1.** Type A. For $m \geqslant 1$ and $i = -2, -1, \ldots, m-1$ we define the moments $\mu^m_{i,A}$ by

$$\mu^m_{i,A} = \int_{-1}^{1} x S_{2m+1}(x)(x^2+\xi^2)^{i+1}(1-x^2)^{\alpha-1}\,dx.$$

Type B. For $m \geqslant 1$ and $i = -2, -1, \ldots, m-1$ we define the moments $\mu^m_{i,B}$ by

$$\mu^m_{i,B} = \int_{-1}^{1} x S_{2m+1}(x)(x^2-\xi^2)^{i+1}(1-x^2)^{\alpha-1}\,dx.$$

Type C. For $m \geqslant 1$ and $i = 0, 1, \ldots, m-1$ we define the moments $\mu^m_{i,C}$ by

$$\mu^m_{i,C} = \int_{-1}^{1} x S_{2m+1}(x)(x^2-1)^{i}\,dx.$$

Type D. For $m \geqslant 1$ and $i = -1, 0, \ldots, m-1$ we define the moments $\mu^m_{i,D}$ by

$$\mu^m_{i,D} = \int_{-1}^{1} x S_{2m+1}(x)(x^2-\xi^2)^{i}(1-x^2)^{\alpha-1}\,dx, \qquad i \geqslant 0,$$

$$\mu^m_{-1,D} = 0.$$

Type E. For $m \geqslant 1$ and $i = -1, 0, \ldots, m-1$ we define the moments $\mu^m_{i,E}$ by

$$\mu^m_{i,E} = \int_{-1}^{1} x S_{2m+1}(x)(x^2+\xi^2)^{i}(1-x^2)^{\alpha-1}\,dx.$$

Note that the moments $\mu^m_{-2,j}$, $j = A, B$, and $\mu^m_{-1,E}$ are finite.

LEMMA 4.4. *For $m \geqslant 2$, $1 \leqslant i \leqslant m-1$, we have*

$$\mu^m_{i,A} = -2\lambda\{\alpha^A_i \mu^m_{i-1,A} - \beta^A_i \mu^m_{i-2,A} + (i-1)\,\gamma^A_i \mu^m_{i-3,A}\},$$

$$\mu^m_{i,B} = 2\lambda\{\alpha^B_i \mu^m_{i-1,B} + \beta^B_i \mu^m_{i-2,B} + (i-1)\,\gamma^B_i \mu^m_{i-3,B}\},$$

*and for $m \geqslant 3$, $2 \leqslant i \leqslant m-1$, we have*

$$\mu^m_{i,C} = 2i\lambda\{(2i+1)\,\mu^m_{i-1,C} + 2(i-1)\,\mu^m_{i-2,C}\},$$

$$\mu^m_{i,D} = 2\lambda\{\alpha^D_i \mu^m_{i-1,D} + \beta^D_i \mu^m_{i-2,D} + (i-2)\,\gamma^D_i \mu^m_{i-3,D}\},$$

$$\mu^m_{i,E} = -2\lambda\{\alpha^E_i \mu^m_{i-1,E} - \beta^E_i \mu^m_{i-2,E} + (i-2)\,\gamma^E_i \mu^m_{i-3,E}\},$$

*where $\alpha^j_i > 0$, $\beta^j_i > 0$ and $\gamma^j_i \geqslant 0$, $j = A, B, D, E$.*

*Proof.* For type A we find, using $\langle S_{2m+1}, x(x^2+\xi^2)^i\rangle_S = 0$ for $m \geqslant 2$ and $1 \leqslant i \leqslant m-1$,

$$\mu_{i,A}^m = -\lambda(2i+1)\int_{-1}^{1} S'_{2m+1}(x)(x^2+\xi^2)^i\,(1-x^2)^\alpha\,dx +$$

$$+2\lambda i\xi^2\int_{-1}^{1} S'_{2m+1}(x)(x^2+\xi^2)^{i-1}\,(1-x^2)^\alpha\,dx.$$

Integration by parts (the constant term vanishes) gives

$$\int_{-1}^{1} S'_{2m+1}(x)(x^2+\xi^2)^i\,(1-x^2)^\alpha\,dx$$

$$= 2(i+\alpha)\int_{-1}^{1} xS_{2m+1}(x)(x^2+\xi^2)^i\,(1-x^2)^{\alpha-1}\,dx$$

$$-2i(1+\xi^2)\int_{-1}^{1} xS_{2m+1}(x)(x^2+\xi^2)^{i-1}\,(1-x^2)^{\alpha-1}\,dx.$$

From this we obtain

$$\alpha_i^A = (2i+1)(i+\alpha)$$
$$\beta_i^A = i((2i+1)+\xi^2(4i+2\alpha-1))$$
$$\gamma_i^A = 2i\xi^2(1+\xi^2).$$

In the same way we find for type D

$$\alpha_i = (i+\alpha-1)(1+2i)$$
$$\beta_i = i(4i\xi^2-5\xi^2+2\alpha\xi^2-2i+1)-(\xi^2-1)$$
$$\gamma_i = 2i\xi^2\,(\xi^2-1).$$

Again by replacing $\xi^2$ by $-\xi^2$ we get the result for type B from type A. In the same way we can obtain the result for type E, from type D. We get the result for type C from type B, by substituting $\xi^2=1$ and $\alpha=0$. ▮

LEMMA 4.5. *For $m \geqslant 1$ we have*

$$\operatorname{sgn}\mu_{i,A}^m = (-1)^{m+i+1}, \qquad -1 \leqslant i \leqslant m-1,$$
$$\mu_{i,B}^m > 0, \qquad -1 \leqslant i \leqslant m-1,$$
$$\mu_{i,C}^m < 0, \qquad 0 \leqslant i \leqslant m-1,$$
$$\mu_{i,D}^m < 0, \qquad 0 \leqslant i \leqslant m-1,$$
$$\operatorname{sgn}\mu_{i,E}^m = (-1)^{m+i+1}, \qquad 0 \leqslant i \leqslant m-1.$$

*Proof.* Type A. For $\mu^m_{-1,A}$ we get, using integration by parts,

$$\mu^m_{-1,A} = \int_{-1}^{1} xS_{2m+1}(x)(1-x^2)^{\alpha-1}\,dx = \frac{1}{2\alpha}\int_{-1}^{1} S'_{2m+1}(x)(1-x^2)^{\alpha}\,dx. \qquad (18)$$

Using $\langle G^{(\alpha)}_{2m}, 1\rangle_1 = 0$ and applying (13) repeatedly, we obtain in the same way as in Lemma 4.2

$$\operatorname{sgn} \mu^m_{-1,A} = (-1)^m.$$

For $\mu^m_{0,A}$ we obtain from $\langle S_{2m+1}, x\rangle_S = 0$,

$$\mu^m_{0,A} = -\lambda \int_{-1}^{1} S'_{2m+1}(x)(1-x^2)^{\alpha}\,dx.$$

From (18) we see that sgn $\mu^m_{0,A}$ is the opposite of sgn $\mu^m_{-1,A}$, hence

$$\operatorname{sgn} \mu^m_{0,A} = (-1)^{m+1}.$$

By Lemma 4.4 and induction the lemma follows for type A. For type B the proof is similar.

Type C. For $\mu^m_{0,C}$ we find, using $\langle S_{2m+1}, x\rangle_S = 0$,

$$\mu^m_{0,C} = -\lambda \int_{-1}^{1} S'_{2m+1}(x)\,dx - 2MS_{2m+1}(1) = -2(\lambda+M)\,S_{2m+1}(1).$$

Since $d^C_{2m} < 0$, we find from (13) and $\langle G^{(0)}_{2m}, 1\rangle_1 = 0$ that

$$\operatorname{sgn} \int_{-1}^{1} S'_{2m+1}(x)\,dx = \operatorname{sgn} \int_{-1}^{1} S'_{2m-1}(x)\,dx$$

and therefore that

$$\operatorname{sgn} S_{2m+1}(1) = \operatorname{sgn} S_1(1).$$

Since $S_1(1) = P_1(1) > 0$, we have

$$\mu^m_{0,C} < 0.$$

For $m \geqslant 2$ we find, using $\langle S_{2m+1}, x(x^2-1)\rangle_S = 0$,

$$\mu^m_{1,C} = -\lambda \int_{-1}^{1} S'_{2m+1}(x)(3x^2-1)\,dx.$$

Then we use $\langle G_{2m}^{(0)}, 3x^2-1\rangle_1 = 0$ and (13) repeatedly, to find in a similar way as in the proof of lemma 4.2

$$\mu_{1,C}^m < 0.$$

Here we used $S_3'/3 = Q_2 - d_2^C$ and the orthogonality of $Q_2$ to determine the sign of $\int_{-1}^1 S_3'(x)(3x^2-1)\,dx$. Now by Lemma 4.4 and induction the lemma follows.

Type D. Using $\langle S_{2m+1}, x(x^2-\xi^2)\rangle_S = 0$ we get for $m \geqslant 2$

$$\begin{aligned}\mu_{1,D}^m &= \lambda \int_{-1}^1 S_{2m+1}'(x)(\xi^2-3x^2)\,d\psi_1\\ &= 3\lambda \int_{-1}^1 S_{2m+1}'(x)(1-x^2)^\alpha\,dx - 2\lambda\xi^2 \int_{-1}^1 S_{2m+1}'(x)\,\frac{(1-x^2)^\alpha}{\xi^2-x^2}\,dx\\ &\quad -4M\xi^2\lambda S_{2m+1}'(\xi).\end{aligned}$$

Integration by parts gives

$$\int_{-1}^1 S_{2m+1}'(x)(1-x^2)^\alpha\,dx = 2\alpha \int_{-1}^1 S_{2m+1}(x)\,x(1-x^2)^{\alpha-1}\,dx.$$

From $\langle S_{2m+1}, x\rangle_S = 0$ we obtain

$$\begin{aligned}-2\lambda \int_{-1}^1 S_{2m+1}'(x)\,\frac{(1-x^2)^\alpha}{\xi^2-x^2}\,dx - 4M\lambda S_{2m+1}'(\xi)\\ = 2\int_{-1}^1 S_{2m+1}(x)\,x(1-x^2)^{\alpha-1}\,dx.\end{aligned}$$

And then we have

$$\mu_{1,D}^m = (6\lambda\alpha + 2\xi^2)\,\mu_{0,D}^m.$$

In a similar way as for type A we find

$$\mu_{0,D}^m < 0.$$

By Lemma 4.4 and induction the lemma then follows for type D. For type E the proof is similar. ∎

Again let $\pi(x)$ denote an even monic polynomial of degree $2k$, such that all zeros of $\pi(x)$ are real. For each type $j$ we define an integral $I_{2,l}^j$ by

$$I^j_{2,l} = \int_{-1}^{1} xS_{2m+1}(x)\,\pi(x)(x^2+\xi^2)^l\,(1-x^2)^{\alpha-1}\,dx, \qquad j=A, E,$$

$$I^j_{2,l} = \int_{-1}^{1} xS_{2m+1}(x)\,\pi(x)(x^2-\xi^2)^l\,(1-x^2)^{\alpha-1}\,dx, \qquad j=B, D,$$

$$I^C_{2,l} = \int_{-1}^{1} xS_{2m+1}(x)\,\pi(x)(x^2-1)^l\,dx.$$

LEMMA 4.6. *Let* $m \geqslant 1$.

(i) *If* $0 \leqslant k+l \leqslant m$, *then* $\operatorname{sgn} I^A_{2,l} = (-1)^{m+k+l}$.

(ii) *If* $0 \leqslant k+l \leqslant m$ *and all zeros of* $\pi(x)$ *lie in the interval* $[-|\xi|, |\xi|]$, *then* $I^B_{2,l} > 0$.

(iii) *If* $0 \leqslant k+l \leqslant m-1$ *and all zeros of* $\pi(x)$ *lie in the interval* $[-1, 1]$, *then* $I^C_{2,l} < 0$.

(iv) *If* $0 \leqslant k+l \leqslant m-1$ *and all zeros of* $\pi(x)$ *lie in the interval* $[-|\xi|, |\xi|]$, *then* $I^D_{2,l} < 0$.

(v) *If* $0 \leqslant k+l \leqslant m-1$, *then* $\operatorname{sgn} I^E_{2,l} = (-1)^{m+k+l+1}$.

*Proof.* The proof is similar to the proof of Lemma 4.3. ▮

## 5. LOCATION OF THE ZEROS

Using Lemmas 4.3 and 4.6 from the previous section we can determine the position of the zeros of $S_n$ for all types, with respect to other (known) polynomials. Knowing the position of the zeros we find that $S_n$ has $n$ different, real zeros for type A, B, C, and D and at least $n-2$ different, real zeros for type E. Since the method by which we determine the position of the zeros, is the same for all the types of symmetrically coherent pairs, we only give the proof for type A.

THEOREM 5.1. *Let* $\{d\psi_0, d\psi_1\}$ *denote a symmetrically coherent pair of type* A. *Let* $n \geqslant 3$. *Then* $S_n$ *has n different, real zeros. Let* $\{G_n^{(\alpha-1)}\}_n$ *denote the MOPS of Gegenbauer polynomials. Let* $g_1 < \cdots < g_m$ *denote the positive zeros of* $G_n^{(\alpha-1)}$ *and let* $s_1 < \cdots < s_m$ *denote the positive zeros of* $S_n$. *Then*

$$g_1 < s_1 < \cdots < g_m < s_m.$$

*Proof.* Note that $-g_i$ is also a zero of $G_n^{(\alpha-1)}$. For $n = 2m$ put

$$\pi(x) = \frac{G_{2m}^{(\alpha-1)}(x)}{x^2 - g_i^2}, \qquad i = 1, 2, ..., m.$$

For $n=2m+1$ put

$$\pi(x)=\frac{G_{2m+1}^{(\alpha-1)}(x)}{x(x^2-g_i^2)}, \qquad i=1,2,...,m.$$

Using Lemma 4.3 for $n=2m$ and Lemma 4.6 for $n=2m+1$, we obtain

$$\int_{-1}^{1} S_n(x)\frac{G_n^{(\alpha-1)}(x)}{x^2-g_i^2}(1-x^2)^{\alpha-1}\,dx<0.$$

Applying Gauss-quadrature on the zeros of $G_n^{(\alpha-1)}$ gives

$$\frac{S_n(g_i)(G_n^{(\alpha-1)})'(g_i)}{g_i}<0.$$

Since $g_i>0$, we now have $\operatorname{sgn} S_n(g_i)=-\operatorname{sgn}(G_n^{(\alpha-1)})'(g_i)$. And since $(G_n^{(\alpha-1)})'$ has opposite sign in two consecutive zeros of $G_n^{(\alpha-1)}$, the same holds for $S_n$. Thus $S_n$ has a zero in each of the intervals $(g_i, g_{i+1})$ and in each of the intervals $(-g_{i+1}, -g_i)$, $i=1,...,m-1$. Since $(G_n^{(\alpha-1)})'(g_m)>0$, we have $S_n(g_m)<0$. Because $S_n$ is monic, $S_n$ has a zero on the right of $g_m$ and therefore also a zero on the left of $-g_m$. Using $S_{2m+1}(0)=0$, we have found $n$ different, real zeros of $S_n$. ▮

*Remark* 5.1. Expanding $G_n^{(\alpha-1)}$ in terms of $P_i^\alpha$ gives

$$G_n^{(\alpha-1)}(x)=P_n^\alpha(x)+D_{n-1}\frac{n}{n-2}P_{n-2}^\alpha(x).$$

Using Lemma 3.2 we then get

$$S_n^\infty(x)=G_n^{(\alpha-1)}(x).$$

This means that the lower bound from Theorem 5.1 cannot be improved.

THEOREM 5.2. *Let $\{d\psi_0, d\psi_1\}$ denote a symmetrically coherent pair of type* A. *Let $\{P_n^\alpha\}_n$ denote the MOPS with respect to $d\psi_0$. Let $n\geqslant 3$, let $p_1<\cdots<p_m$ denote the positive zeros of $P_n^\alpha$ and let $s_1<\cdots<s_m$ denote the positive zeros of $S_n$. Then*

$$s_1<p_1<\cdots<s_m<p_m.$$

*Proof.* For $n=2m$ put

$$\pi(x)=\frac{P_{2m}^\alpha(x)}{x^2-p_i^2}(x^2+\xi^2), \qquad i=1,2,...,m.$$

For $n=2m+1$ put

$$\pi(x)=\frac{P^\alpha_{2m+1}(x)}{x(x^2-p_i^2)}\,(x^2+\xi^2),\qquad i=1,2,\ldots,m.$$

Using Lemma 4.3 for $n=2m$ and Lemma 4.6 for $n=2m+1$, we obtain

$$\int_{-1}^{1} S_n(x)\,\frac{P^\alpha_n(x)}{x^2-p_i^2}\,(x^2+\xi^2)(1-x^2)^{\alpha-1}\,dx>0.$$

Applying Gauss-quadrature on the zeros of $P^\alpha_n$ gives

$$\frac{S_n(p_i)(P^\alpha_n)'\,(p_i)}{p_i}>0.$$

Using arguments similar to those in the proof of Theorem 5.1 gives the desired result. ∎

*Remark* 5.2. Observe that if $\lambda$ tends to zero, then $S_n$ tends to $P^\alpha_n$. Therefore the upper bounds from Theorem 5.2 cannot be improved.

THEOREM 5.3. *Let $\{d\psi_0, d\psi_1\}$ denote a symmetrically coherent pair of type* B. *Then $S_n$ has n different, real zeros. Let $\{P^\alpha_n\}_n$ denote the MOPS with respect to $d\psi_0$. Let $n\geqslant 3$, let $p_1<\cdots<p_m$ denote the positive zeros of $P^\alpha_n$ and let $s_1<\cdots<s_m$ denote the positive zeros of $S_n$. Then*

$$p_1<s_1\cdots<p_m<s_m.$$

*Remark* 5.3. As for type A observe that if $\lambda$ tends to zero, then $S_n$ tends to $P^\alpha_n$. Therefore the lower bounds from theorem 5.3 cannot be improved.

THEOREM 5.4. *Let $\{d\psi_0, d\psi_1\}$ denote a symmetrically coherent pair of type* B. *Let $n\geqslant 3$. Let $\{G^{(\alpha-1)}_n\}_n$ denote the MOPS of Gegenbauer polynomials. Let $g_1<\cdots<g_m$ denote the positive zeros of $G^{(\alpha-1)}_n$ and let $s_1<\cdots<s_m$ denote the positive zeros of $S_n$. Then*

$$s_1<g_1<\cdots<s_m<g_m.$$

*Remark* 5.4. As for type A we can prove for type B that

$$S^\infty_n(x)=G^{(\alpha-1)}_n(x).$$

This means that the upper bound from Theorem 5.4 cannot be improved.

*Remark* 5.5. Notice that, different from type A, for type B we first compare the zeros of $S_n$ to the zeros of $P_n$ (i.e., the polynomials orthogonal with respect to $d\psi_0$) and then to $Q_n$ (i.e., the polynomials with respect to $d\psi_1$). This is because we need a lower bound for the zeros in order to prove that $S_n$ has $n$ different, real zeros.

THEOREM 5.5. *Let $\{d\psi_0, d\psi_1\}$ denote a symmetrically coherent pair of type* C. *Then $S_n$ has n different, real zeros. Let $\{G_n^{(0)}\}_n$ denote the MOPS of Legendre polynomials. Let $n \geqslant 3$, let $g_1 < \cdots < g_m$ denote the positive zeros of $G_n^{(0)}$ and let $s_1 < \cdots < s_m$ denote the positive zeros of $S_n$. Then*

$$g_1 < s_1 \cdots < g_m < s_m.$$

THEOREM 5.6. *Let $\{d\psi_0, d\psi_1\}$ denote a symmetrically coherent pair of type* D. *Let $n \geqslant 3$. Then $S_n$ has n different, real zeros. Let $\{G_n^{(\alpha-1)}\}_n$ denote the MOPS of Gegenbauer polynomials. Let $g_1 < \cdots < g_m$ denote the positive zeros of $G_n^{(\alpha-1)}$ and let $s_1 < \cdots < s_m$ denote the positive zeros of $S_n$. Then*

$$g_1 < s_1 < \cdots < g_m < s_m.$$

*Remark* 5.6. If $\lambda$ tends to zero, then $S_n$ tends to $G_n^{(\alpha-1)}$. This means that the lower bounds from Theorem 5.6 cannot be improved.

THEOREM 5.7. *Let $\{d\psi_0, d\psi_1\}$ denote a symmetrically coherent pair of type* E. *Let $n \geqslant 3$. Then $S_n$ has at least $n-2$ different, real zeros. Let $\{G_n^{(\alpha-1)}\}_n$ denote the MOPS of Gegenbauer polynomials. Let $g_1 < \cdots < g_m$ denote the positive zeros of $G_n^{(\alpha-1)}$ and let $s_2 \leqslant \ldots \leqslant s_m$ denote the largest positive zeros of $S_n$. Then*

$$g_1 < s_2 < \cdots < s_m < g_m.$$

*If $S_n$ has n different, real zeros, let $s_1$ denote the smallest positive zero. Then*

$$s_1 < g_1.$$

*Proof.* In the same way as in the proof of Theorem 5.1 we find

$$\frac{S_n(g_i)(G_n^{(\alpha-1)})'(g_i)}{g_i} > 0. \tag{19}$$

Since $g_i > 0$, we now have $\operatorname{sgn} S_n(g_i) = \operatorname{sgn}(G_n^{(\alpha-1)})'(g_i)$. And since $(G_n^{(\alpha-1)})'$ has opposite sign in two consecutive zeros of $G_n^{(\alpha-1)}$, the same holds for $S_n$. Thus $S_n$ has a zero in each of the intervals $(g_i, g_{i+1})$ and in each of the intervals $(-g_{i+1}, -g_i)$, $i = 1, \ldots, m-1$. Using $S_{2m+1}(0) = 0$, we have found $n-2$ different, real zeros of $S_n$.

Suppose that that $S_n$ has two zeros in the interval $(g_k, g_{k+1})$, $1 \leqslant k \leqslant m-1$. Then, since $\operatorname{sgn} S_n(g_k) = \operatorname{sgn} (G_n^{(\alpha-1)})'(g_i)$, $S_n$ must have three zeros in the interval $(g_k, g_{k+1})$. Because $S_n$ is an odd or an even polynomial, $S_n$ then would have $n+2$ zeros. Because $S_n$ is monic and $(G_n^{(\alpha-1)})'(g_m) > 0$, $S_n$ cannot have a zero in $(g_m, \infty)$, because then $S_n$ would have two zeros in $(g_m, \infty)$ and two zeros in $(-\infty, -g_m)$. Therefore, if $S_n$ has $n$ positive zeros, then $s_1 < g_1$. ∎

*Remark* 5.7. If $\lambda$ tends to zero, then $S_n$ tends to $G_n^{(\alpha-1)}$. Therefore the upper bounds from Theorem 5.7 cannot be improved.

Because Theorem 5.7 states that $S_n$ has at least $n-2$ different, real zeros, the theorem does not exclude the possibility that $S_n$ has *complex* zeros. In the next section we proof that under certain conditions $S_n$ indeed has complex zeros. But first we proof that the odd polynomials $S_{2m+1}$ of type E have $2m+1$ *real* zeros.

LEMMA 5.1. *Let $\{d\psi_0, d\psi_1\}$ denote a symmetrically coherent pair of type E. Let $m \geqslant 1$, then $S_{2m+1}$ has $2m$ different extremata. Let $\{Q_n^\alpha\}_n$ denote the MOPS with respect to $d\psi_1$, let $q_1 < \cdots < q_m$ denote the positive zeros of $Q_{2m}^\alpha$ and let $\sigma_1 < \cdots < \sigma_m$ denote the positive extremata of $S_{2m+1}$. Then*

$$q_1 < \sigma_1 < \cdots < q_m < \sigma_m.$$

*Proof.* Consider for $0 \leqslant i \leqslant m$ the following integral

$$M_i^m = \int_{-1}^{1} S'_{2m+1}(x)(x^2+\xi^2)^{i-1}(1-x^2)^\alpha \, dx.$$

For $i = 0$ we get from $\langle S_{2m+1}, x\rangle_S = 0$

$$M_0^m = -\frac{\mu_{0,E}^m}{\lambda}.$$

For $1 \leqslant i \leqslant m$ integration by parts shows

$$M_i^m = (2i+2\alpha-2)\,\mu_{i-1,E}^m - 2(i-1)(\xi^2+1)\,\mu_{i-2,E}^m.$$

Then by lemma 4.5

$$\operatorname{sgn} M_i^m = (-1)^{m+i}. \tag{20}$$

Let $\pi(x)$ denote an even monic polynomial of degree $2k$, $1 \leqslant k \leqslant m$, such that all zeros of $\pi(x)$ are real. Let $x_1, ..., x_k$ denote the nonnegative zeros of $\pi(x)$. For $1 \leqslant i \leqslant k$ put $t_i^2 = x_i^2 + \xi^2$. Then

$$\begin{aligned}\pi(x) &= (x^2+\xi^2-t_1^2)\cdots(x^2+\xi^2-t_k^2)\\ &= \sum_{i=0}^{k} c_i(x^2+\xi^2)^i,\end{aligned}$$

where $c_k = 1$ and $\operatorname{sgn} c_i = (-1)^{k-i}$.

Since

$$\int_{-1}^{1} S'_{2m+1}(x)\,\pi(x)\,\frac{(1-x^2)^\alpha}{x^2+\xi^2}\,dx = \sum_{i=0}^{k} c_i M_i^m,$$

we obtain from (20)

$$\operatorname{sgn}\int_{-1}^{1} S'_{2m+1}(x)\,\pi(x)\,\frac{(1-x^2)^\alpha}{x^2+\xi^2}\,dx = (-1)^{m+k}. \tag{21}$$

For $1 \leqslant i \leqslant m$ put

$$\pi(x) = \frac{Q_{2m}^\alpha(x)}{x^2-q_i^2}.$$

Then (21) gives

$$\int_{-1}^{1} S'_{2m+1}(x)\,\frac{Q_{2m}^\alpha(x)}{x^2-q_i^2}\,\frac{(1-x^2)^\alpha}{x^2+\xi^2}\,dx < 0.$$

Applying Gauss-quadrature on the zeros of $Q_{2m}^\alpha$ gives

$$\frac{S'_{2m+1}(q_i)(Q_{2m}^\alpha)'(q_i)}{q_i} < 0. \tag{22}$$

Then arguments similar to those in the proof of theorem 5.1 give the desired result. ∎

THEOREM 5.8. *Let* $\{d\psi_0, d\psi_1\}$ *denote a symmetrically coherent pair of type* E. *Let* $m \geqslant 0$, *then* $S_{2m+1}$ *has* $2m+1$ *different, real zeros.*

*Proof.* We prove for $m = 2k$ that $S_{2m+1}$ has a zero in $(0, g_1)$, where $g_1$ is the smallest positive zeros of $G_{4k+1}^{(\alpha-1)}$. From Theorem 5.7 then follows that $S_{4k+1}$ has exactly $4k+1$ different, real zeros. For $m = 2k+1$ the proof runs along the same lines.

From (22) we can determine $\operatorname{sgn} S'_{4k+1}(q_1)$, where $q_1$ is the smallest positive zero of $Q^{\alpha}_{2m}$:

$$(Q^{\alpha}_{4k})'(q_1) < 0 \Rightarrow S'_{4k+1}(q_1) > 0.$$

Since $S'_{4k+1}$ is an even polynomial, we also have $S'_{4k+1}(-q_1) > 0$. According to Lemma 5.1 $S_{4k+1}$ has no extremata in the interval $(-q_1, q_1)$, hence

$$S'_{4k+1}(0) > 0. \tag{23}$$

From (19) we get

$$(G^{(\alpha-1)}_{4k+1})'(g_1) < 0 \Rightarrow S_{4k+1}(g_1) < 0.$$

This combined with (23) shows that $S_{4k+1}$ must have a zero in the interval $(0, g_1)$. ∎

5.1. *Complex zeros*

We consider the Gegenbauer-Sobolev polynomials of type E. If $\lambda$ tends to zero, then $S_n$ tends to $G^{(\alpha-1)}_n$. Thus $S_n$ can only have complex zeros if $\lambda$ is sufficiently large. Therefore we consider the case where $\lambda \to \infty$.

From Lemma 3.2 we have

$$S^{\infty}_n(x) = G^{(\alpha-1)}_n(x) + D_{n-1}\frac{n}{n-2}G^{(\alpha-1)}_{n-2}(x), \qquad n \geqslant 3. \tag{24}$$

LEMMA 5.2. *$S^{\infty}_{2m}$ has complex zeros if and only if*

$$D_{2m-1} > \frac{(\alpha+m-\frac{3}{2})(m-1)(m-\frac{1}{2})}{m(\alpha+2m-\frac{5}{2})(\alpha+2m-\frac{3}{2})}, \quad m \geqslant 2.$$

*Proof.* $S^{\infty}_{2m}$ has complex zeros if and only if $\operatorname{sgn} S^{\infty}_{2m}(0)$ is opposite to $\operatorname{sgn} G^{(\alpha-1)}_{2m}(0)$. From (24) we obtain

$$S^{\infty}_{2m}(0) = G^{(\alpha-1)}_{2m}(0) + \frac{m}{m-1}D_{2m-1}G^{(\alpha-1)}_{2m-2}(0).$$

From the recurrence relation (6) we find

$$G^{(\alpha-1)}_{2m}(0) = -\frac{(2m-1)(2\alpha+2m-3)}{(2\alpha+4m-5)(2\alpha+4m-3)}G^{(\alpha-1)}_{2m-2}(0).$$

Then

$$S_{2m}^{\infty}(0)=G_{2m}^{(\alpha-1)}(0)\left(1-\frac{m(2\alpha+4m-5)(2\alpha+4m-3)}{(m-1)(2m-1)(2\alpha+2m-3)}\,D_{2m-1}\right).$$

Hence, $S_{2m}^{\infty}$ has complex zeros if and only if

$$D_{2m-1}>\frac{(\alpha+m-\frac{3}{2})(m-1)(m-\frac{1}{2})}{m(\alpha+2m-\frac{5}{2})(\alpha+2m-\frac{3}{2})}.$$

This proves the lemma. ∎

LEMMA 5.3. *$S_{2m}^{\infty}$ has complex zeros if and only if*

$$I_m^{\alpha}(\xi)>\frac{(2m-1)(\alpha+m-\frac{3}{2})}{m(m+\alpha-\frac{1}{2})},\quad m\geqslant 2,$$

*where*

$$I_m^{\alpha}(\xi)=\frac{\displaystyle\int_{-1}^{1}\frac{(1-t)^{m+\alpha-1}\,(1+t)^{m-\frac{1}{2}}}{(t+1+2\xi^2)^m}\,dt}{\displaystyle\int_{-1}^{1}\frac{(1-t)^{m+\alpha-2}\,(1+t)^{m-\frac{3}{2}}}{(t+1+2\xi^2)^{m-1}}\,dt}.$$

*Proof.* Expanding $Q_n^{\alpha}$ in terms of $G_i^{(\alpha)}$, gives

$$Q_n^{\alpha}(x)=G_n^{(\alpha)}(x)+D_nG_{n-2}^{(\alpha)}(x).$$

Using $\langle Q_{2m-1}^{\alpha},x\rangle_1=0$, we then obtain

$$D_{2m-1}=-\frac{\displaystyle\int_{-1}^{1}G_{2m-1}^{(\alpha)}(x)\,\frac{x(1-x^2)^{\alpha}}{x^2+\xi^2}\,dx}{\displaystyle\int_{-1}^{1}G_{2m-3}^{(\alpha)}(x)\,\frac{x(1-x^2)^{\alpha}}{x^2+\xi^2}\,dx}. \tag{25}$$

From (8) we get

$$\begin{aligned}J_m^{\alpha}&=\int_{-1}^{1}G_{2m-1}^{(\alpha)}(x)\,\frac{x(1-x^2)^{\alpha}}{x^2+\xi^2}\,dx\\&=2^{1-m}\int_{-1}^{1}P_{m-1}^{(\alpha,\frac{1}{2})}(2x^2-1)\,\frac{x^2\,(1-x^2)^{\alpha}}{x^2+\xi^2}\,dx.\end{aligned}$$

The substution $2x^2-1:=t$ gives

$$J_m^\alpha = 2^{-(m+\alpha-\frac{1}{2})} \int_{-1}^{1} P_{m-1}^{(\alpha,\frac{1}{2})}(t)\, \frac{(1-t)^\alpha (1+t)^{\frac{1}{2}}}{t+1+2\xi^2}\, dt.$$

Using the Rodrigues formula (9) and integration by parts $m-1$ times gives

$$J_m^\alpha = (-1)^{m-1} \frac{(m-1)!}{2^{m+\alpha-\frac{1}{2}}} \frac{\Gamma(m+\alpha+\frac{1}{2})}{\Gamma(2m+\alpha-\frac{1}{2})} \int_{-1}^{1} \frac{(1-t)^{m+\alpha-1}(1+t)^{m-\frac{1}{2}}}{(t+1+2\xi^2)^m}\, dt.$$

From (25) we then obtain

$$D_{2m-1} = \frac{(m+\alpha-\frac{1}{2})(m-1)}{2(2m+\alpha-\frac{3}{2})(2m+\alpha-\frac{5}{2})} \frac{\displaystyle\int_{-1}^{1} \frac{(1-t)^{m+\alpha-1}(1+t)^{m-\frac{1}{2}}}{(t+1+2\xi^2)^m}\, dt}{\displaystyle\int_{-1}^{1} \frac{(1-t)^{m+\alpha-2}(1+t)^{m-\frac{3}{2}}}{(t+1+2\xi^2)^{m-1}}\, dt}.$$

From Lemma 5.2 the lemma then follows. ∎

THEOREM 5.9. *Let $m \geqslant 2$. If $|\xi|$ is sufficiently small, then $S_{2m}^\infty$ has complex zeros.*

*Proof.* We use Lemma 5.3. Substiting $t:=2x-1$ gives

$$I_m^\alpha(\xi) = 2 \frac{\displaystyle\int_0^1 \frac{(1-x)^{m+\alpha-1} x^{m-\frac{1}{2}}}{(x+\xi^2)^m}\, dx}{\displaystyle\int_0^1 \frac{(1-x)^{m+\alpha-2} x^{m-\frac{3}{2}}}{(x+\xi^2)^{m-1}}\, dx}.$$

For $\xi \to 0$ we get

$$\lim_{\xi\to 0} I_m^\alpha(\xi) = \frac{2B(m+\alpha,\frac{1}{2})}{B(m+\alpha-1,\frac{1}{2})} = \frac{2(m+\alpha-1)}{m+\alpha-\frac{1}{2}},$$

where $B$ denotes the beta function. It's clear that the condition

$$\frac{2(m+\alpha-1)}{m+\alpha-\frac{1}{2}} > \frac{(2m-1)(m+\alpha-\frac{3}{2})}{m(m+\alpha-\frac{1}{2})}$$

is satisfied for all $m$ and $\alpha$. ∎

THEOREM 5.10. *If $m$ is sufficiently large, then $S_{2m}^{\infty}$ has no complex zeros.*

*Proof.* We will determine the asymptotic behaviour of $I_m^{\alpha}(\xi)$ for $m \to \infty$. We will use Laplace's method (see, e.g., [9]). The numerator of $I_m^{\alpha}(\xi)$ we call $N_m^{\alpha}(\xi)$ and the denominator $D_m^{\alpha}(\xi)$. Then

$$N_m^{\alpha}(\xi) = \int_{-1}^{1} e^{-mp(t)} q_N(t)\, dt,$$

$$D_m^{\alpha}(\xi) = \int_{-1}^{1} e^{-mp(t)} q_D(t)\, dt,$$

where

$$p(t) = -\log(1-t) - \log(1+t) + \log(t+1+2\xi^2),$$

$$q_N(t) = (1-t)^{\alpha-1}(1+t)^{-\frac{1}{2}},$$

$$q_D(t) = \frac{(1-t)^{\alpha-2}(1+t)^{-\frac{3}{2}}}{(t+1+2\xi^2)^{-1}}.$$

The function $p(t)$ has a minimum in

$$t_0 = -1 - 2\xi^2 + 2\sqrt{\xi^2+\xi^4} \in (-1, 1).$$

Then Laplace's method gives

$$I_m^{\alpha}(\xi) \sim \frac{q_N(t_0)}{q_D(t_0)} = \frac{(1-t_0)(1+t_0)}{t_0+1+2\xi^2} < 2, \qquad m \to \infty.$$

Now by Lemma 5.3 we find that $S_{2m}^{\infty}$ has no complex zeros if $m \to \infty$. ∎

## REFERENCES


1. M. G. de Bruin, W. G. M. Groenevelt, and H. G. Meijer, Zeros of Sobolev orthogonal polynomials of Hermite type, *Appl. Math. Comput.*, in press.
2. M. G. de Bruin and H. G. Meijer, Zeros of Sobolev orthogonal polynomials following from coherent pairs, *J. Comput. Appl. Math.*, in press.
3. A. Iserles, P. E. Koch, S. P. Nørsett, and J. M. Sanz-Serna, On polynomials orthogonal with respect to certain Sobolev inner products, *J. Approx. Theory* **65** (1991), 151–175.
4. F. Marcellán, M. Alfaro, and M. L. Rezola, Orthogonal polynomials on Sobolev spaces: Old and new directions, *J. Comput. Appl. Math.* **48** (1993), 113–131.
5. F. Marcellán, T. E. Pérez, and M. A. Piñar, Laguerre–Sobolev orthogonal polynomials, *J. Comput. Appl. Math.* **71** (1996), 245–265.
6. F. Marcellán, T. E. Pérez, and M. A. Piñar, Gegenbauer–Sobolev orthogonal polynomials, *in* "Nonlinear Numerical Methods and Rational Approximation II, Proceedings" (A. Cuyt, Ed.), pp. 71–82, Kluwer Academic, Dordrecht, 1994.

7. H. G. Meijer, A short history of orthogonal polynomials in a Sobolev space. I. The nondiscrete case, *Nieuw Arch. Wisk. (4)* **14** (1996), 93–112.
8. H. G. Meijer, Determination of all coherent pairs, *J. Approx. Theory* **89** (1997), 321–343.
9. F. W. J. Olver, "Asymptotics and Special Functions," Academic Press, New York/London, 1974.
10. G. Szegő, "Orthogonal Polynomials," 4th ed., Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, 1975.